Nonexchangebility and Radial Asymmetry Identification
via Bivariate Quantiles and Financial Applications

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A Class of Bivariate Probability Integral Transforms

Let $H_a$ and $H_b$ be bivariate distribution functions with common continuous marginal distributions $F(x) = P(X \leq x)$ and $G(y) = P(Y \leq y)$.

Let $H_b(x, y) = P(X \leq x, Y \leq y)$. Consider the random variable $H_a(X,Y)$ with distribution function

$$(H_a|H_b)(t) = P(H_a(X,Y) \leq t) = \mu_{H_b}\{(x, y) \in \mathbb{R}^2|H_a(x, y) \leq t\}, t \in [0,1].$$

Theorem. [Nelsen et al. (2001)] Let $H_a, H_b, F, G, X$ and $Y$ be as above and $C_a$ and $C_b$ be the underlying copulas of $H_a$ and $H_b$, respectively. Then

$$(H_a|H_b)(t) = (C_a|C_b)(t), \ t \in [0,1].$$
Kendall Distribution

When $H_a = H_b = H$ with copula $C$ we get the Kendall distributed random variable $H(X, Y)$, or equivalently, $C(U, V)$, defined by

$$K(t) = P(C(U, V) \leq t) = P(H(X, Y) \leq t), \quad t \in [0,1].$$

Remark: The Kendall distribution is univariate characteristic of bivariate dependence.

Theorem. [Nelsen et al. (2003)] Let $C$ be a bivariate copula and $M = \text{Min}(u, v)$ and $W = \text{Max}(u + v - 1,0)$. If $K(t) = t$, for all $t \in [0,1]$, then $C = M$, and if $K(t) = 1$ for all $t \in [0,1]$, then $C = W$.

Theorem. [Capéraà et al. (1997)]

$$0 \leq t \leq K(t) \leq 1.$$
Bivariate Quantile Curves

The quantile of order $p \in [0,1]$ of a random variable $X$ is usually defined by

$Q_X(p) = F_X^{-1}(p) = \text{Inf}\{x: F_X(x) \geq p\}$.

Belzunce et al. (2007) suggest a bivariate version of quantile curves as follows.

Let $(X, Y)$ be a random vector and $(x, y)$ be a point in $\mathbb{R}^2$. We denote by $H_\varepsilon(x, y)$, $\varepsilon \in \{--, -+, +-, ++, +\}$, the following probabilities:

- $H_{--}(x, y) = P(X \leq x, Y \leq y); \quad H_{++}(x, y) = P(X > x, Y > y)$;
- $H_{-+}(x, y) = P(X \leq x, Y > y); \quad H_{+-}(x, y) = P(X > x, Y \leq y)$.

**Definition.** The $p$-th bivariate quantile set, or quantile curve, for the direction $\varepsilon \in \{--, -+, +-, ++, +\}$, is defined as

$Q_{(X,Y)}(p, \varepsilon) = \{(x, y) \in \mathbb{R}^2: H_\varepsilon(x, y) = p\}$. 
Example. Bivariate Quantile Curves [Belzunce et al. (2007)]

Let $(X, Y)$ be a random vector with independent components which are unit exponentially distributed.

For a given $p \in [0,1]$, the quantile curves are the solutions of equations $H_\varepsilon(x, y) = p$ for all $\varepsilon \in \{--,-+,+--,++\}$. The quantile curves are shown in the following figure:
Central Region

**Definition.** Let \((X, Y)\) be a continuous random vector and \(p \in \left(\frac{1}{2}, 1\right]\). The **central region** of order \(p\) is defined by

\[
\Omega_{(X,Y)}(p) = \{(x, y) \in \mathbb{R}^2 : H_\varepsilon(x, y) < p, \quad \forall \varepsilon \in \{--, --, +-, ++\}\}
\]

**Remark.** The central region \(\Omega_{(X,Y)}(p)\) corresponds to the points in the plane which accumulate a probability less than \(p\) in all quadrants.

Figure: The central region.
Lateral Regions

Definition. Let \((X, Y)\) be a continuous random vector and \(p \in [0, 1]\). The lateral region of order \(p\) in the direction \(\varepsilon \in \{--, --, +-, ++, +++, +--\}\) is defined by

\[
L_{(X,Y)}(p, \varepsilon) = \{(x, y) \in \mathbb{R}^2 : H_\varepsilon(x, y) > p\}.
\]

Theorem. [Belzunce et al. (2007)] Let \((X, Y)\) be a continuous random vector with copula \(C\). Then the accumulated probabilities in the central and lateral regions of order \(p\), i.e. \(P[(X, Y) \in \Omega_{(X,Y)}(p)]\) and \(P[(X, Y) \in L_{(X,Y)}(p, \varepsilon)]\), depend only on the copula \(C\).
Main Result

**Theorem 1.** Let \((X, Y)\) be a continuous random vector with copula \(C\) and let \(C^{--}, C^{-+}, C^{+-}\) and \(C^{++}\) be the copulas

\[
C^{--}(u, v) = C(u, v); \quad C^{-+}(u, v) = u - C(u, 1 - v);
\]
\[
C^{+-}(u, v) = v - C(1 - u, v) \quad \text{and} \quad C^{++}(u, v) = u + v - 1 + C(1 - u, 1 - v).
\]

Then the following equalities hold:

\[
P[(X, Y) \in L_{(X,Y)}(p, --)] = 1 - K^{--}(p); \quad P[(X, Y) \in L_{(X,Y)}(p, --)] = 1 - K^{-+}(p);
\]
\[
P[(X, Y) \in L_{(X,Y)}(p, + -)] = 1 - K^{+-}(p); \quad P[(X, Y) \in L_{(X,Y)}(p, + +)] = 1 - K^{++}(p).
\]

**Remark:** The functions \(K^{--}(p), K^{-+}(p), K^{+-}(p)\) and \(K^{++}(p)\) are Kendall distribution functions of the random variables \(H(X, Y), F(X) - H(X, Y), G(Y) - H(X, Y)\) and \(1 - F(X) - G(Y) + H(X, Y)\), respectively.
Nonexchangeability and Radial Asymmetry Identification

**Theorem 2.** Let \((X, Y)\) be a continuous random vector with copula \(C\). If exchangeability holds, i.e. \(C(u, v) = C(v, u)\), for all \((u, v)\) in \([0,1]^2\), then

\[
K^{--}(p) = K^{++}(p), \quad \text{for all } p \in [0,1].
\]

**Theorem 3.** Let \((X, Y)\) be a continuous random vector with copula \(C\). If radial symmetry holds, i.e. \(C(u, v) = 1 - (1 - u) - (1 - v) + C(1 - u, 1 - v)\) for all \((u, v)\) in \([0,1]^2\), then

\[
K^{--}(p) = K^{++}(p) \text{ and } K^{+-}(p) = K^{+-}(p), \quad \text{for all } p \in [0,1].
\]
New Measures of Nonexchangeability and Radial Asymmetry

**Definition.** The **nonexchangeability measure** of the bivariate continuous distribution with copula $C$ is defined by:

$$
\Pi_C = 3 \int_0^1 [K^{++}(p) - K^{+-}(p)]^2 dp.
$$

The exchangeability implies $\Pi_C = 0$. The inverse statement is false, see Example 3.

**Definition.** The **radial asymmetry measure** of the bivariate continuous distribution with copula $C$ is defined by:

$$
\Sigma_C = Max(\Sigma, \Pi_C),
$$

with $\Sigma = 3 \int_0^1 [K^{--}(p) - K^{+-}(p)]^2 dp$.

The radial symmetry implies $\Sigma_C = Max(\Sigma, \Pi_C) = 0$, i.e. both $\Sigma$ and $\Pi_C$ should be zero.

**Remark.** The measures $\Pi_C$ and $\Sigma_C$ lie in $[0,1]$. 
Example 1. Fréchet-Mardia Family of Copulas

Let $C_{\alpha,\beta}(u, v)$ be the copula given by:

$$C_{\alpha,\beta}(u, v) = \alpha W(u, v) + (1 - \alpha - \beta)uv + \beta M(u, v),$$

where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$.

Since $C_{\alpha,\beta}(u, v) = C_{\alpha,\beta}(v, u)$ and by the radial symmetry of the copulas $W, M$ and $\Pi(u, v) = uv$, it follows that $\Pi_C = \Sigma_C = 0.$
Example 2. Maximally Nonexchangeable and Radial Asymmetric Copula

Let \( C(u, v) \) be the copula given by

\[
C(u, v) = \text{Min}\left(u, v, \left(u - \frac{2}{3}\right)^+ + \left(v - \frac{1}{3}\right)^+\right).
\]

This copula corresponds to the maximally nonexchangeable and radially asymmetric according to the measures proposed by Nelsen (2007).

Kendall distributions related to copula \( C(u, v) \) are given by

\[
K^{--}(p) = \text{Min}\left(p, \frac{1}{3}\right) + \text{Min}\left(p, \frac{2}{3}\right) = K^{++}(p),
\]

\[
K^{+-}(p) = \frac{1 + 2\mathbb{1}(p \geq \frac{1}{3})}{3} \quad \text{and} \quad K^{-+}(p) = \frac{2 + \mathbb{1}(p \geq \frac{2}{3})}{3}.
\]

Evaluating the integrals, one gets \( \Sigma = 0 \) and \( \Pi_C = \frac{2}{9} \). Hence, \( \Sigma_C = \frac{2}{9} \).
Example 3. $C(u, v) \neq C(v, u)$, but $\Pi_C = 0$

Let $C(u, v)$ be a copula of “Shuffle of $M$” type, see Mikusinski et al. (1992), having its support as represented in the following figure:

![Diagram of copula]

It is easy to see that $C(u, v) \neq C(v, u)$. However, the copula $C(u, v)$ is radially symmetric and using theorem 3 it follows that $K^{+-}(p) = K^{-+}(p)$, for all $p \in [0,1]$, and $\Sigma_C = \text{Max}(\Sigma, \Pi_C) = 0$, i.e. $\Pi_C = 0$. 
Empirical Kendall Distribution

Let \( \tilde{C} \left( \frac{i}{n} , \frac{j}{n} \right) \) be the empirical copula of the sample vector \( \{(x_k, y_k)\}_{k=1}^n \) of the continuous random vector \((X, Y)\), given by

\[
\tilde{C} \left( \frac{i}{n} , \frac{j}{n} \right) = \frac{1}{n} \{\text{number of points } (x_k, y_k) \text{ such that } x_k \leq x_{(i)} \text{ and } y_k \leq y_{(j)}\},
\]

where \((i)\) denotes the \(i\)-th order statistics.

Note that the empirical Kendall distribution is given by

\[
\tilde{K}(p) = \frac{1}{n} \left\{ \text{number of pairs } (x_k, y_k) \text{ whose ranks } (i, j) \text{ are such that } \tilde{C} \left( \frac{i}{n} , \frac{j}{n} \right) \leq p \right\}.
\]
Empirical Versions of the Measures $\Pi_C$ and $\Sigma_C$

**Theorem 4.** The sample versions of $\Pi_C$ and $\Sigma_C$, for a sample of size $n$ of pairs $(x_k, y_k)$, are given by:

$$\bar{\Pi}_C = \bar{\Sigma}_2 \text{ and } \bar{\Sigma}_C = Max(\bar{\Sigma}_1, \bar{\Sigma}_2),$$

where

$$\bar{\Sigma}_1 = 3 \sum \frac{1}{n} \left[ 1 - \frac{i}{n} - \frac{j}{n} + \tilde{C}(\frac{i}{n}, \frac{j}{n}) - \tilde{C}(\frac{n-i}{n}, \frac{n-j}{n}) \right]^2 \text{ and }$$

$$\bar{\Sigma}_2 = 3 \sum \frac{j}{n} - \tilde{C}(\frac{n-i}{n}, \frac{j}{n}) - \frac{i}{n} + \tilde{C}(\frac{i}{n}, \frac{n-j}{n})^2.$$

(the sums above are over the $n$ points in the sample)
Exchangeability Hypothesis Test – Notations

Let \( \{x_{1k}, y_{1k}\}_{k=1}^{n} \) and \( \{x_{2k}, y_{2k}\}_{k=1}^{m} \) be two samples of the continuous random vector \( (X, Y) \), of sizes \( n \) and \( m \), respectively, and let \( H_{n}(x, y) \) and \( H_{m}(x, y) \) be the corresponding empirical bivariate distribution functions with empirical margins \( F_{n,X}(x) \) and \( F_{m,X}(x) \) of \( X \), and \( G_{n,Y}(y) \) and \( G_{m,Y}(y) \) of \( Y \).

Additionally, let

\[
T_{n} = F_{n,X}(x_{1k}) - H_{n}(x_{1k}, y_{1k}) \quad \text{and} \quad T_{m} = G_{m,Y}(y_{2k}) - H_{m}(x_{2k}, y_{2k}).
\]

The quantities \( T_{n} \) and \( T_{m} \) are random variables and let their distribution functions be \( A_{T_{n}}(p) \) and \( B_{T_{m}}(p) \), respectively, for \( p \in (0,1) \).
Exchangeability Hypothesis Test

The hypothesis test with given significance level $\alpha$ for verifying the exchangeability may be defined in the following manner:

$$\mathcal{H}_0: A_{T_n}(p) = B_{T_m}(p), \text{ for all } p \in (0,1),$$

against

$$\mathcal{H}_1: A_{T_n}(p) \neq B_{T_m}(p), \text{ for at least one value of } p \in (0,1).$$

\textbf{Remark 1.} The exchangeability implies the equality between $K^{+-}(p)$ and $K^{-+}(p)$, for all $p \in (0,1)$, what leads to the structure of null hypothesis $\mathcal{H}_0$.

\textbf{Remark 2.} A traditional methodology for verifying the validity of the null hypothesis $\mathcal{H}_0$ (equality of two distribution functions) is the two-sample Kolmogorov-Smirnov test.

\textbf{Remark 3.} A similar approach may be used to test the radial symmetry.
Simulation Example

Let \((U, V)\) be a random vector with uniform in \((0, 1)\) margins and with copula

\[
C_{\alpha, \beta}(u, v) = uv + uv(1 - u)(1 - v)[\alpha + (\beta - \alpha)v(1 - u)],
\]

where \(|\alpha| \leq 1\) and \((1/2)[\alpha - 3 - (9 + 6\alpha - 3\alpha^2)^{1/2}] \leq \beta \leq 1\).

The copula \(C_{\alpha, \beta}(u, v)\) is nonexchangeable for \(\alpha \neq \beta\). When \(\alpha = \beta\), \(C_{\alpha, \beta}(u, v)\) belongs to the Farlie-Gumbel-Morgenstern family of copulas with parameter \(\alpha\).

We simulated 10000 pairs \((U, V)\) with \(\alpha = \beta = 1\) [i.e. \(C_{1, 1}(u, v) = C_{1, 1}(v, u)\)].

We get the following results:

- The empirical value \(\hat{\Pi}_C = 0.0000061\) is close to 0, in concordance to the exchangeability of \(C_{1, 1}(u, v)\).
- The Kolmogorov-Smirnov test for identity of distributions \(K^{-+}(p)\) and \(K^{+-}(p)\) produced a p-value 0.3305, just confirming that \(C_{1, 1}(u, v)\) is exchangeable.
Simulation Example: Related Graphics

Simulation of $(U,V)$ with Copula FGM $(1, 1)$

QQ-Plot $K_+ \times K_+$

Histogram of $K_+$

Histogram of $K_-$
Financial Application

We investigate the exchangeability and radial symmetry of the copula of the log-returns of the currency parities EUR x USD and USD x BRL. The data covers the period 09/08/1999 – 08/08/2009.

A graphical representation of the data is given below.
Financial Application – The Empirical Central and Lateral Regions

In order to make visual check of the symmetry of the central and lateral regions we built their empirical graphical representation for \( p = 0.55 \) and \( p = 0.7 \). The results are given below.
Financial Application – Related Graphics

The histograms and QQ-Plots of the variables used in testing the exchangeability and radial symmetry of the copula are shown below.
Financial Application – Hypothesis Test

The graphics presented previously suggest that the copula of log-returns of the currency parities EUR x USD and USD x BRL is both exchangeable and radial symmetric.

Performing the proposed hypothesis test we get the following results:

- The Kolmogorov-Smirnov test for identity of distributions $K^{-+}(p)$ and $K^{+-}(p)$ produced a p-value $0.9116$, just confirming the exchangeability of the copula.

- The Kolmogorov-Smirnov test for identity of distributions $K^{--}(p)$ and $K^{++}(p)$ produced a p-value $0.2104$, confirming the radial symmetry of the copula.

These results show that the variables analyzed have a strong symmetry in their dependence structure, what have important consequences in market risk computation, for instance.
Open Problems

- To specify additional conditions under which the proposed measures $\Pi_C$ and $\Sigma_C$ are equal to zero iff the copula is symmetric or radially symmetric respectively;

- To obtain best-possible bounds for the proposed measures;

- To get characterizations of the families of copulas that are maximally asymmetric with respect to these measures;

- Power study of the proposed hypothesis test.
References


