MONOTONICITY AND THE AUMANN-SHAPLEY COST-SHARING METHOD IN THE DISCRETE CASE

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ABSTRACT. We give an axiomatization of the Aumann-Shapley cost-sharing method in the discrete case by means of monotonicity and no merging or splitting (Sprumont, 2005 [16]). Monotonicity has not yet been employed to characterize this method in such a case, unlike the case in which goods are perfectly divisible, for which Monderer and Neyman (1988) [10] and Young (1985) [19] characterize the Aumann-Shapley price mechanism. We also offer two variations of our main axiomatization.

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Key words: Cost sharing, Aumann-Shapley method, monotonicity.

1. INTRODUCTION

In this paper we provide several axiomatizations for the Aumann-Shapley cost-sharing method in the discrete case by employing monotonicity properties. Aumann-Shapley pricing is a method to solve cost-sharing problems in which agents consume a quantity of possibly different goods. If a finite set \( N = \{1, ..., n\} \) denotes the set of agents \( i \), \( q_i \) is the quantity of good \( i \) demanded by agent \( i \) and \( q \) is the corresponding consumption profile, we want to split \( C(q) \) among the agents. We may price goods instead of sharing the cost \( C(q) \). Both approaches are equivalent, and we have chosen the latter, following Sprumont [16].

In the discrete case agents can consume several units of the good. The consumption \( q_i \) of agent \( i \) is a non-negative integer which indicates how many units of good \( i \) are consumed by agent \( i \). The simplest cost-sharing method to solve such problems is the Shapley-Shubik method introduced by Shubik [14]. Given a coalition \( S \subseteq N \) of agents, Shubik considers the cost of the consumptions of these agents: \( v(S) = C(q_S,0_{N\setminus S}) \); and calculates the Shapley value [13] of the transferable utility game \( v \) with \( N \) as set of players. This Shapley value gives us precisely the assignment by the Shapley-Shubik method. The Aumann-Shapley method is
also defined by means of a transferable utility game, but now it is taken into account more information about cost. Indeed, costs of intermediate consumptions of agents are regarded and not just those associated with consumptions \( q_i \). Given an agent \( i \), each unit consumed by \( i \) is viewed as a different player. So there is a set \( N_i \) of \( |N_i| = q_i \) players and the grand set \( N^q = \bigcup_{i \in N} N_i \). Therefore we have coalitions of players, which are subsets of \( N^q \). Given \( S \subseteq N^q \), \( x(S) = (|S \cap N_i|)_{i \in N} \) represents the units demanded by agents in \( S \) and the cost associated with these demands is \( v^q(S) = C(x(S)) \). The Shapley value of \( v^q \) gives allocations for players in \( N^q \). Since each agent \( i \) is represented by the elements in \( N_i \), the sum of the allocations of these elements is by definition the allocation for agent \( i \) provided by the Aumann-Shapley method.


In the discrete case, the Aumann-Shapley method has been characterized in two works. In the first one, due to Calvo and Santos [4], the method is characterized by means of balanced contributions, as Myerson [11] does for the Shapley value, and requiring agents who consume nothing to pay zero. In the second one Sprumont [16] employs additivity, dummy and no merging or splitting. Dummy is the natural extension for cost-sharing problems of the dummy axiom for transferable utility games. It requires that dummies get zero. An agent \( i \) is dummy if the cost of any consumption profile does not change if \( i \) consumes one more unit. No merging or splitting, which is described in detail in Section 3, prevents splitting manipulations. According to it, if an agent \( i \) splits into several agents (and therefore splits his consumption into several consumptions), the split agents have to pay in sum the same quantity as agent \( i \) had to. The characterization by Sprumont [16] can be seen as formed by basic axioms, like the one given by Mirman and Tauman [7] for the Aumann-Shapley price mechanism with perfectly divisible goods and Shapley’s [13] characterization of his value.

If we look at other characterizations for the Aumann-Shapley price mechanism with perfectly divisible goods we find that monotonicity has also been employed,
as Young [18] does when he characterizes the Shapley value. Monotonicity for cost-sharing problems states that if the marginal costs associated with an agent at all consumption levels decrease, then the cost share corresponding to that agent cannot increase. Young [19] and Monderer and Neyman [10] obtain two characterizations for the Aumann-Shapley price mechanism using this axiom. However, nothing on this has been done in the discrete case. And that is the aim of this paper. We provide an axiomatization for the Aumann-Shapley method with monotonicity. We prove that this method is the only one that satisfies monotonicity and no merging or splitting. So we prove that additivity and dummy can be replaced by monotonicity in Sprumont [16]. Observe that the same happens for Young’s [18] characterization regarding Shapley’s [13] characterization of his value. Moreover, as we explain with more details in Section 3, if we look at the characterizations of the Aumann-Shapley price mechanism given by Young [19] and by Monderer and Neyman [10] we realize that they employ stronger axioms than no merging or splitting.

We have to say that, like Sprumont [16], we deal with non-decreasing cost functions, unlike Young [19] and Monderer and Neyman [10], who use cost functions which are not necessarily non-decreasing. In addition, as Sprumont [16] does, we suppose that cost shares are non-negative. If we considered any cost function we would characterize the Aumann-Shapley method just asking cost shares to be non-negative if the cost function is non-decreasing.

Finally, as Sprumont [16] does, we provide two variations of the axiomatization. No merging or splitting relates cost-sharing problems with different sets of agents. Sprumont [16] changes the axiom in order to relate cost-sharing problems with equal set of agents. In our second characterization we consider this variation. And in the third one we let agents consume bundle of goods.

The paper is structured as follows. After this Introduction, in Section 2 we give preliminaries, in Section 3 we get the main axiomatization, in Section 4 its variations, and the paper ends with conclusions in Section 5 and references.

2. Preliminaries

Most of the definitions and notation here follow those in Sprumont [16].

First let us give some notation. Denote by \( \mathbb{N} \) the set of non-negative integers, and by \( \mathcal{N} \) the set of non-empty finite subsets of \( \mathbb{N} \). If \( N \in \mathcal{N} \), denote by \( |N| \) the cardinality of \( N \), and let \( \mathbb{N}^N \) be the cartesian product of \( \mathbb{N} \) with itself \(|N|\) times. Coordinates of elements in \( \mathbb{N}^N \) are indexed by the elements of \( N \). If \( x = (x_i)_{i \in N} \in \mathbb{N}^N \) and \( M \subseteq N \) we write \( x(M) \) for \( \sum_{i \in M} x_i \), and \( x_M \in \mathbb{N}^M \) denotes the restriction of \( x \) to \( M \). If \( x, y \in \mathbb{N}^N \) we write \( x \leq y \) to denote \( x_i \leq y_i \). Some distinguished
vectors in $\mathbb{N}^N$ are $0 = (0, \ldots, 0)$ and for every $M \subseteq N$ its indicator $1^M$, defined by $1^M_i = 1$ if $i \in M$ and $1^M_i = 0$ otherwise.

A pair $(N, q)$ defines a consumption profile, where $N \in \mathcal{N}$ is the set of agents and $q \in \mathbb{N}^N$ is the list of their consumptions. If $q, q' \in \mathbb{N}^N$, write $[q, q']$ for the set $\{x \in \mathbb{N}^N : q \leq x \leq q'\}$. For each consumption profile $(N, q)$, denote by $\mathcal{C}(N, q)$ the set of cost functions for $(N, q)$, that is non-decreasing functions $C : [0, q] \subset \mathbb{N}^N \to \mathbb{R}_+$. The set of all problems is denoted by $\mathcal{P}$. Then a cost-sharing problem is a triple $(N, q, C)$, where $(N, q)$ is a consumption profile and $C \in \mathcal{C}(N, q)$. The set of all problems is then denoted by $\mathcal{P}$. A cost-sharing method is a mapping $\phi$ that assigns to every cost-sharing problem $(N, q, C) \in \mathcal{P}$ a vector $\phi(N, q, C) \in \mathbb{R}^N_+$ satisfying the budget balance condition: $\sum_{i \in N} \phi_i(N, q, C) = C(q)$, that is, exactly shares all the costs.

Let us formalize the Aumann-Shapley method. Given $(N, q, C) \in \mathcal{P}$, let $\{N_i\}_{i \in N}$ be a family of pairwise disjoint sets such that $|N_i| = q_i$, and let $N^q = \bigcup_{i \in N} N_i$. For each $S \subseteq N^q$ define the demand vector $x(S) = (|S \cap N_i|)_{i \in N}$ and consider the cooperative game $(N^q, v^q)$ defined by $v^q(S) = C(x(S))$. Denote the Shapley value of this game by $\phi^S_h(N^q, v^q)$. The Aumann-Shapley method assigns to every $(N, q, C) \in \mathcal{P}$ the vector $\phi^{A\!h}(N, q, C)$ defined for all $i \in N$ by

$$\phi^k_i(N, q, C) = \sum_{j \in N_i} \phi^S_j(N^q, v^q).$$

Notice that by the anonymity of the Shapley value the real numbers $\phi^S_j(N^q, v^q)$ are independent of the choice of the sets $N_i$, and consequently the Aumann-Shapley method is well defined.

3. An axiomatization of the Aumann-Shapley method

Our first axiomatization is based in two axioms.

Let $(N, q, C) \in \mathcal{P}$, $i \in N$, and $x \in [0, q]$ such that $x_i < q_i$. Then we write $\partial_i C(x) = C(x + 1^i) - C(x)$.

**Monotonicity:** Let $(N, q, C), (N, q, C') \in \mathcal{P}$, and $i \in N$. If for all $x \in [0, q]$ such that $x_i < q_i$ it holds $\partial_i C(x) \leq \partial_i C'(x)$, then $\phi_i(N, q, C) \leq \phi_i(N, q, C')$.

Cost functions $C$ and $C'$ tell us which is the cost associated with different consumption levels corresponding to agents $i$ in $N$ or equivalently production levels of goods $i$. Fixing $i \in N$, if $\partial_i C(x) \leq \partial_i C'(x)$ at all feasible consumption levels,

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1. This is the only difference with the model of Sprumont [16], since this author considers that the domain of the cost function $C$ is all $\mathbb{N}^N$, instead of $[0, q]$.

2. We write $1^i$ instead of the more cumbersome $1^{(i)}$. 
then we can say that when moving from $C'$ to $C$ an improvement in production efficiency has taken place with respect to good $i$. Monotonicity requires that $i$ cannot be penalized in this case, that is, the cost share associated with $i$ cannot increase. So, if efficiency with respect to a particular agent increases, then the cost share associated with that agent cannot increase.

The continuous version of monotonicity (joint with other axioms) is employed by Young [19] and by Monderer and Neyman [10] to characterize the Aumann-Shapley price mechanism. In the continuous version the first partial derivative of the cost function is employed, the one sided first partial derivative on the boundary of the domain. Notice that with both prices and cost shares monotonicity is stated in the same manner since it involves only one agent $i$ and therefore equality $\phi_i(N, q, C) \leq \phi_i(N, q, C')$ holds in both cases.

The second axiom we employ is the one introduced by Sprumont [16] to characterize the Aumann-Shapley cost sharing method in the discrete case.

**No Merging or Splitting:** Let $(N, q, C) \in \mathcal{P}$, $i \in N$, and $I \in \mathcal{N}$ be such that $N \cap I = \{i\}$. Consider a consumption profile $(N', q')$, where $N' = (N \setminus i) \cup I$, $q'$ is such that $q'_j = q_j$ for all $j \in N \setminus i$ and $\sum_{i' \in I} q'_{i'} = q_i$. Define $C' \in \mathcal{C}(N', q')$ by $C'(x) = C\left(x_{N \setminus i}, \sum_{i' \in I} x_{i'}\right)$. Then

$$\sum_{i' \in I} \phi_{i'}(N', q', C') = \phi_i(N, q, C).$$

This axiom avoids splitting manipulations. It relates two problems with different sets of agents. An agent "splits" into different agents whose total consumption equals the consumption of the original one. So agent $i$ splits into agents in $I$ and consumption of $i$ equals consumption of agents in $I$. The other consumptions do not change. Accordingly cost function does not change since the variable associated with agent $i$ has split into the variables associated with the split agents. The sum of those variables plays the role of the variable associated with the original agent $i$.

No merging or splitting requires the cost share of the original agent to be equal to the sum of the cost shares of split agents. Observe that this axiom does not impose any requirement to the cost shares of the remaining agents. Notice also that when $i$ splits into several agents then agent $i$ is also present. So the consumption of $i$ is divided in several quantities and it is supposed that these quantities are associated with $i$ and some new agents.

As Sprumont [16] writes, the counterpart of no merging or splitting for price mechanisms with perfectly divisible goods appears in Tauman [17] as Axiom 5". There are two characterizations of the Aumann-Shapley price mechanism with perfectly divisible goods by means of stronger axioms than Axiom 5". Both characterizations are given for cost functions which are not necessarily increasing. On the one
hand, Young [19] characterizes the Aumann-Shapley price mechanism by employing monotonicity and aggregation invariance. Aggregation invariance, first introduced by Billera and Heath [3], implies symmetry (prices do not depend on the names of the agents), rescaling (if the demands are rescaled so the prices are) and Axiom 5’, but those three axioms do not imply aggregation invariance (Monderer and Neyman [10], Chapter 15). Later on, Monderer and Neyman [10] employ monotonicity, rescaling and consistency to give another characterization of the Aumann-Shapley price mechanism. Consistency, introduced by Mirman and Neyman [6], is weaker than aggregation invariance, and is equivalent to Axiom 5” and symmetry. We point out also that consistency implies that when an agent splits into several agents then the cost shares of the other agents do not change.

Therefore, if we compare our characterization to the above ones, we prove in the present paper that in the discrete case and considering non-decreasing cost functions if we just ask for the non-negativity of the cost-sharing method we can drop out symmetry and rescaling from the characterization introduced by Monderer and Neyman [10]. Notice also that in the discrete case rescaling would have had meaning only when production is represented by non-negative integers.

**Theorem 1.** The Aumann-Shapley method is the only cost-sharing method that satisfies monotonicity and no merging or splitting.

To prove this theorem we need some previous lemmas and further notation.

**Lemma 1.** Let $\phi$ be a cost-sharing method. If $(N, q, C) \in \mathcal{P}$ is such that $C(x) = 0$ for every $x \in [0, q]$, then $\phi_i(N, q, C) = 0$ for every $i \in N$.

**Proof.** By budget balance condition and the non-negativity of $\phi$. 

If $(N, q, C) \in \mathcal{P}$ we write $C_{N\setminus i}(x) = C(x, 0_i)$ for all $x \in [0, q_{N\setminus i}]$. We prove in the following lemma that no merging or splitting implies the following property.

**Zero Independence:** Let $(N, q, C) \in \mathcal{P}$, and $i \in N$. If $q_i = 0$ then $\phi_i(N, q, C) = 0$ and $\phi_{N\setminus i}(N, q, C) = \phi(N\setminus i, q_{N\setminus i}, C_{N\setminus i})$.

That is, if an agent does not require anything he does not pay anything and the others pay as if the former agent were not present. It is a very natural and known axiom. See, for example, Moulin and Shenker [9] or Sprumont [15].

**Lemma 2.** If $\phi$ satisfies no merging or splitting, then $\phi$ satisfies zero independence.

**Proof.** Let $(N, q, C) \in \mathcal{P}$ and $i \in N$ such that $q_i = 0$.

Assume first that $|N| = n \geq 3$. Let $j \in N\setminus i$, and write $I = \{i, j\}$. Applying no merging or splitting to $(N\setminus i, q_{N\setminus i}, C_{N\setminus i})$ and agent $j$, we have

$$
\phi_j(N\setminus i, q_{N\setminus i}, C_{N\setminus i}) = \phi_i((N\setminus i) \setminus j) \cup I, q, C) + \phi_j((N\setminus i) \setminus j) \cup I, q, C),
$$
Since \( ((N\setminus i) \setminus j) \cup I = N \), we can conclude
\[
\phi_j(N \setminus i, q_{N \setminus i}, C_{N \setminus i}) = \phi_i(N, q, C) + \phi_j(N, q, C) \quad \text{for all } j \in N \setminus i,
\] (1)
That implies
\[
\sum_{j \in N \setminus i} \phi_j(N \setminus i, q_{N \setminus i}, C_{N \setminus i}) = (n - 1)\phi_i(N, q, C) + \sum_{j \in N \setminus i} \phi_j(N, q, C),
\]
Then by the definition of \( C_{N \setminus i} \) and budget balance condition we have
\[
C(q) = C_{N \setminus i}(q_{N \setminus i}) = (n - 1)\phi_i(N, q, C) + C(q) - \phi_i(N, q, C).
\]
Hence \((n - 2)\phi_i(N, q, C) = 0\), and consequently \(\phi_i(N, q, C) = 0\). Furthermore from expression (1) it follows \(\phi_j(N, q, C) = \phi_j(N \setminus i, q_{N \setminus i}, C_{N \setminus i})\) for all \(j \in N \setminus i\) as it was to be proved.

Observe that the case \(|N| = 1\) is trivial, so assume that \(N = \{i, j\}, i \neq j\), and w.l.o.g. \(q_i = 0\). Let \(\ell \notin N\), and write \(I = \{i, \ell\}\), \(N' = (N \setminus i) \cup I\), and define \(q' = (q, 0_\ell)\) and \(C' \in \mathcal{C}(N', q')\) by \(C'(x) = C(x_N)\) for all \(x \in [0, q']\). Then \(|N'| = 3\) and we can apply the result of the first part of the proof. Since \(q'_i = q_i = 0\), we have \(\phi_i(N', q', C') = 0\). On the other hand \(q'_\ell = 0\), hence \(\phi_i(N', q', C') = \phi_i(N, q, C)\). Both equalities together give \(\phi_i(N, q, C) = 0\). In addition from budget balance we have \(\phi_j(N, q, C) = \phi_j(N \setminus i, q_{N \setminus i}, C_{N \setminus i})\) and the proof is complete. \(\square\)

Let \(\pi : N \to N\) be a bijection. If \(N \in \mathcal{N}\), write \(\pi N = \{\pi(i) : i \in N\}\). If \(x \in \mathbb{R}_+^N\), define \(\pi x \in \mathbb{R}_+^N\) by \((\pi x)_{\pi(i)} = x_i\) for all \(i \in N\). Moreover if \((N, q)\) is a consumption profile and \(C \in \mathcal{C}(N, q)\), define \(\pi C \in \mathcal{C}(\pi N, \pi q)\) by \(\pi C(\pi x) = C(x)\). Finally write \(\pi(N, q, C) = (\pi N, \pi q, \pi C)\).

Sprumont [16] shows that no merging or splitting implies the following property.

**Weak Symmetry:** Let \((N, q, C) \in \mathcal{P}, i \in N\), and \(\pi : N \to N\) a bijection. If \(i \in N\) is such that \(\pi(j) = j\) for all \(j \in N \setminus i\), then \(\phi_{\pi(i)}(N, q, C) = \phi_i(N, q, C)\).

According to this axiom, if an agent is renamed his cost share does not change. Notice that weak symmetry does not say anything with respect to the agents whose name is not modified.

**Lemma 3** (Sprumont [16]). If \(\phi\) satisfies no merging or splitting then \(\phi\) satisfies weak symmetry.

Next lemma, which is proved by Sprumont [16], will also be used in this section. Let \(M \in \mathcal{N}\) such that \(|M| = 2s - 1\) for some positive integer \(s\). Define \(\mathcal{M}(s) = \{R \subseteq M : |R| = s\}\) and \(\mathcal{M}(s - 1) = \{R' \subseteq M : |R'| = s - 1\}\). Notice that
\( R' \in \mathcal{M}(s - 1) \) if and only if \( M \setminus R' \in \mathcal{M}(s) \), and hence \( |\mathcal{M}(s - 1)| = |\mathcal{M}(s)| \). For every \( R \in \mathcal{M}(s) \) define the vector \( u^R \in \mathbb{R}^{\mathcal{M}(s - 1)} \) by

\[
u^R(R') = \begin{cases} 
1, & \text{if } R' \subset R; \\
0, & \text{otherwise.}
\end{cases}
\] (2)

**Lemma 4** (Sprumont [16]). The vectors \( u^R \) form a basis for \( \mathbb{R}^{\mathcal{M}(s - 1)} \).

The following lemma deals with symmetric agents. Given \((N, q, C) \in \mathcal{P}\) we say that two agents \( i, j \in N \) are symmetric if for any bijection \( \pi : \mathbb{N} \rightarrow \mathbb{N} \) such that \( \pi(i) = j, \pi(j) = i \), and \( \pi(k) = k \) for every \( k \in N \setminus \{i, j\} \) it holds \( \pi(N, q, C) = (N, q, C) \).

**Lemma 5.** Let \((T \cup S, q, C) \in \mathcal{P}\) such that \( T \cap S = \emptyset \) and the agents in \( S \) are symmetric. If \( \phi \) satisfies no merging or splitting, and for every bijection \( \pi : \mathbb{N} \rightarrow \mathbb{N} \) such that \( \pi(k) = k \) for every \( k \in T \) it holds \( \sum_{k \in T} \phi_k \pi(T \cup S, q, C) = \sum_{k \in T} \phi_k (T \cup S, q, C) \), then \( \phi_i (T \cup S, q, C) = \phi_j (T \cup S, q, C) \) for every \( i, j \in S \).

**Proof.** Let \( M \in \mathbb{N} \) such that \( S \subseteq M, M \cap T = \emptyset \) and \( |M| = 2s - 1 \). Define \( \mathcal{M}(s) \) as before Lemma 4 and write \( \mu(s) = |\mathcal{M}(s)| \). For every \( R \in \mathcal{M}(s) \) let \( \pi^R \) be a bijection of \( \mathbb{N} \) such that \( \pi^R S = R \), and \( \pi^R(k) = k \) for every \( k \in T \). Consider the problem \( \pi^R(T \cup S, q, C) \) (notice that this problem is independent of the particular choice of \( \pi^R \) since the agents in \( S \) are symmetric). By budget balance it follows that

\[
\sum_{i \in R} \phi_i \pi^R(T \cup S, q, C) = C(q) - \sum_{k \in T} \phi_k \pi^R(T \cup S, q, C)
\] (3)

for all \( R \in \mathcal{M}(s) \). Define \( \mathcal{M}(s - 1) \) also as before Lemma 4. Now for any \( R' \in \mathcal{M}(s - 1) \), by Lemma 3, there exists a number \( \gamma(R') \) such that

\[
\phi_i (T \cup R' \cup i, \pi^{R' \cup i} q, \pi^{R' \cup i} C) = \gamma(R')
\]

for all \( i \in M \setminus R' \). Hence expression (3) can be rewritten as

\[
\sum_{i \in R} \gamma(R' \setminus i) = C(q) - \sum_{k \in T} \phi_k \pi^R(T \cup S, q, C)
\]

for all \( R \in \mathcal{M}(s) \). This is a system of \( \mu(s) \) linear equations in the \( \mu(s - 1) = \mu(s) \) variables \( \gamma(R') \), \( R' \in \mathcal{M}(s - 1) \) that can be rewritten in the form

\[
u^R \cdot \gamma = C(q) - \sum_{k \in T} \phi_k \pi^R(T \cup S, q, C)
\]

for all \( R \in \mathcal{M}(s) \), where \( \gamma = (\gamma(R')) \in \mathbb{R}^{\mathcal{M}(s - 1)} \) and \( u^R \in \mathbb{R}^{\mathcal{M}(s - 1)} \) is defined in expression (2). Since

\[
\sum_{k \in T} \phi_k \pi(T \cup S, q, C) = \sum_{k \in T} \phi_k (T \cup S, q, C),
\]

clearly

\[
\gamma(R') = \frac{1}{s} \left( C(q) - \sum_{k \in T} \phi_k (T \cup S, q, C) \right)
\]
for every $R'$ is a solution. Moreover, by Lemma 4 the system has a unique solution. Then for all $i \in S$, choosing $R' = S \setminus i$ yields

$$
\phi_i(T \cup S, q, C) = \frac{1}{s} \left( C(q) - \sum_{k \in T} \phi_k(T \cup S, q, C) \right),
$$

and the conclusion follows. \qed

In the next two proofs we use the fact that every cost function $C \in \mathcal{C}(N, q)$ can be expressed as a linear combination of primitive cost functions $C_r \in \mathcal{C}(N, q)$ ($r \in [0, q], r \neq 0$). That is, $C = \sum_{0 \neq r \leq q} a_r C_r$, where $a_r \in \mathbb{R}$ and

$$
C_r(x) = \begin{cases} 
1 & \text{if } x \geq r, \\
0 & \text{otherwise}.
\end{cases}
$$

Given such an expression $C = \sum_{0 \neq r \leq q} a_r C_r$, for each $\rho \in \mathbb{R}$, $1 \leq \rho \leq q(N)$ (remind $q(N) = \sum_{i \in N} q_i$), define $\alpha_\rho = \max_{0 \neq r \leq q} a_r$, and $\tilde{\alpha}_r = \alpha_{r(N)} - \alpha_r \geq 0$. Also let $\tilde{C} = \sum_{0 \neq r \leq q} \alpha_r(N) C_r$, that is a cost function (obviously it is non-decreasing) for which all the agents with the same consumption demand are symmetric, and in particular those whose demand is 1. We can write $C = \tilde{C} - \sum_{0 \neq r \leq q} \tilde{\alpha}_r C_r$.

Consider all such expressions for $C$ in which $\tilde{C}$ is symmetric for agents with demand 1, and $\tilde{\alpha}_r \geq 0$ for all $r$. Define the index $\iota$ of $\tilde{C}$ to be the minimum number of terms which appear in the sum of such an expression.

**Lemma 6.** If $\phi$ is a cost-sharing method that satisfies monotonicity and no merging or splitting, and $(N, q, C) \in \mathcal{P}$ be such that $q \leq 1^N$, then $\phi(N, q, C) = \phi^{\text{ASH}}(N, q, C)$.

**Proof.** Let $(N, q, C) \in \mathcal{P}$ be such that $q \leq 1^N$. By Lemma 2 and since $\phi^{\text{ASH}}$ satisfies zero independence, we may assume that $q = 1^N$. The lemma will be proved by induction on the index $\iota$.

If $\iota = 0$ then all the agents in $N$ are symmetric, so from Lemma 5, for the case $T = \emptyset$ and $S = N$, it follows $\phi(N, q, C) = \phi^{\text{ASH}}(N, q, C)$.

Assume now that $\phi$ coincides with $\phi^{\text{ASH}}$ whenever the index of $\tilde{C}$ is at most $\iota$, and let $C$ have index $\iota + 1$ with expression $C = \tilde{C} - \sum_{k=1}^{\iota+1} \tilde{\alpha}_k C_{r_k}$.

Let $S = \{i \in N : (r_k)_i = 1 \text{ for every } k = 1, \ldots, \iota + 1\}$ and assume that $i \notin S$. Define the cost function $D = \tilde{C} - \sum_{k:(r_k)_i = 1} \tilde{\alpha}_k C_{r_k}$. The index of $D$ is at most $\iota$ and $\partial_i D(x) = \partial_i C(x)$ for all $x \in [0, q]$ s. t. $x_i < q_i$, so by induction and monotonicity it follows that

$$
\phi_i(N, q, C) = \phi_i(N, q, D) = \phi_i^{\text{ASH}}(N, q, D) = \phi_i^{\text{ASH}}(N, q, C).
$$

It remains to show that $\phi_i(N, q, C) = \phi_i^{\text{ASH}}(N, q, C)$ when $i \in S$. But notice that the agents in $S$ are symmetric in $(N, q, C)$. Moreover since we have proved that $\phi_i(N, q, C) = \phi_i^{\text{ASH}}(N, q, C)$ when $i \in N \setminus S$, we can apply Lemma 5 and
obtain \( \phi_i(N, q, C) = \phi_j(N, q, C) \) for every \( i, j \in S \). By budget balance the proof is complete.

And this is the proof of Theorem 1.

**Proof.** It is clear that \( \phi^{A_{Sh}} \) satisfies no merging or splitting and monotonicity. Conversely, let \( \phi \) be a cost-sharing method that satisfies these two properties.

For any \( k = 0, 1, \ldots \), let \( \mathcal{D}(k) = \{(N, q, C) \in \mathcal{P} : |\{i \in N : q_i > 1\}| \leq k\} \), i.e. the set of problems where no more than \( k \) agents demand several units. To prove this theorem we have to show that \( \phi(N, q, C) = \phi^{A_{Sh}}(N, q, C) \) for every \( (N, q, C) \in \mathcal{D}(k) \) and for every \( k = 0, 1, 2, \ldots \).

We will proceed by induction on \( k \). From Lemma 6 it follows that \( \phi(N, q, C) = \phi^{A_{Sh}}(N, q, C) \) for every \( (N, q, C) \in \mathcal{D}(0) \).

So let \( k \geq 0 \) be fixed and assume that \( \phi(N, q, C) = \phi^{A_{Sh}}(N, q, C) \) for every \( (N, q, C) \in \mathcal{D}(k) \). Now let \( (N, q, C) \in \mathcal{D}(k+1) \) and let us prove that \( \phi(N, q, C) = \phi^{A_{Sh}}(N, q, C) \).

By the induction hypothesis we can assume that \( (N, q, C) \in \mathcal{D}(k+1) \setminus \mathcal{D}(k) \), i.e. exactly \( k+1 \) agents demand several units.

So consider a problem \( (N, q, C) \in \mathcal{D}(k+1) \). By Lemma 2, we may also assume that \( q \geq 1^N \).

**STEP 1:** \( \phi_i(N, q, C) = \phi_i^{A_{Sh}}(N, q, C) \) for every \( i \in N \) such that \( q_i \geq 2 \). Indeed, let \( I \in \mathcal{N} \) such that \( N \cup I = \{i\} \) and \( |I| = q_i \), and consider a consumption profile \( (N', q') \), where \( N' = (N \setminus i) \cup I \), \( q' \) is such that \( q'_j = q_j \) for all \( j \in N \setminus i \) and \( q'_j = 1 \) for all \( j \in I \). Define \( C' \in \mathcal{C}(N', q') \) by \( C'(x) = C(x_{N \setminus i}, \sum_{i' \in I} x_{i'}) \). Then by No Merging or Splitting, the induction hypothesis and the definition of \( \phi^{A_{Sh}} \) we get

\[
\phi_i(N, q, C) = \sum_{i' \in I} \phi_{i'}(N', q', C') = \sum_{i' \in I} \phi_{i'}^{A_{Sh}}(N', q', C') = \phi_i^{A_{Sh}}(N, q, C).
\]

**STEP 2:** \( \phi_i(N, q, C) = \phi_i^{A_{Sh}}(N, q, C) \) for every \( i \in N \) such that \( q_i = 1 \). Denote \( N_1 = \{i \in N : q_i = 1\} \). We will prove that \( \phi_i(N, q, C) = \phi_i^{A_{Sh}}(N, q, C) \) by induction on the index \( i \) of \( C \).

If \( i = 0 \) then all the agents in \( N_1 \) are symmetric. Then by step 1 we can apply Lemma 5 and the conclusion follows.

Assume now that \( \phi \) coincides with \( \phi^{A_{Sh}} \) whenever the index of \( C \) is at most \( i \), and let \( C \) have index \( i + 1 \) with expression \( C = \tilde{C} - \sum_{k=1}^{i+1} \alpha_{rk} C_{rk} \).

Let \( S = \{i \in N_1 : (r_k)_{i} = 1 \) for every \( k = 1, \ldots, i + 1\} \) and assume that \( i \notin S \). Define the cost function \( D = \tilde{C} - \sum_{k=1}^{i+1} \alpha_{rk} C_{rk} \). The index of \( D \) is at most \( i \) and \( \partial D(x) = \partial_i C(x) \) for all \( x \in [0, q] \) s. t. \( x_i < q_i \), so by induction and monotonicity
it follows that
\[ \phi_i(N, q, C) = \phi_i(N, q, D) = \phi_i^{Ash}(N, q, D) = \phi_i^{Ash}(N, q, C). \]

To finish the proof it remains to show that \( \phi_i(N, q, C) = \phi_i^{Ash}(N, q, C) \) when \( i \in S \). But the agents in \( S \) are symmetric in \((N_1, q, C)\), and taking into account step 1, and that we have proved that \( \phi_i(N, q, C) = \phi_i^{Ash}(N, q, C) \) when \( i \in N_1 \setminus S \), the proof is complete by Lemma 5.

**Remark 1.** If we consider cost functions which are not necessarily non-decreasing (obviously cost shares might be non-negative), we have to add another axiom to obtain the axiomatization. It suffices to require cost shares to be non-negative if the cost function is non-decreasing. Notice that Lemma 1 holds and the rest too. Furthermore, we do not need \( \widetilde{C} \) for the proofs, just the expression \( C = \sum_{0 \neq r \leq q} a_r C_r \).

**Remark 2.** If the domain of cost functions \( C \in \mathcal{C}(N, q) \) is \( \mathbb{N}^N \) instead of \([0, q]\), then if we ask the axiom which states \( \phi(N, q, C) = \phi(N, q, C') \) when \( C = C' \) on \([0, q]\), we have the same results, writing monotonicity accordingly.

### 4. Variations

Spumon [16] gives a characterization for the Aumann-Shapley method where he considers a variation of non merging or splitting for which the set of agents is fixed. We need some notation for that.

Given a consumption profile \((N, q)\) and \( S \subseteq N \), define
\[ C^1_S(N, q) = \left\{ C \in \mathcal{C}(N, q) : \text{there exists } c \text{ such that } C(z) = c(z_{N \setminus S}, \sum_{i \in S} z_i) \right\}, \]

that is the cost functions for which the goods consumed by agents in \( S \) are perfect substitutes. If \( q, q' \in \mathbb{N}^N \) and \( S \subseteq N \) satisfy \( \sum_{i \in S} q_i = \sum_{i \in S} q'_i \) and \( q_{N \setminus S} = q'_{N \setminus S} \), and \( C \in C^1_S(N, q) \), then there exists \( C' \in C^1_S(N, q') \) which has associated the same \( c \) corresponding to \( C \). Though the domains of \( C \) and \( C' \) are different we can say that both cost functions coincide since goods consumed by agents in \( S \) are perfect substitutes.\(^3\) It holds \( C = C' \) on \([0, q] \cap [0, q']\).

**No Reshuffling:** For all \( q, q' \in \mathbb{N}^N \), all \( S \subseteq N \), and all \( C \in C^1_S(N, q) \),
\[ \sum_{i \in S} q_i = \sum_{i \in S} q'_i \text{ and } q_{N \setminus S} = q'_{N \setminus S} \text{ imply } \sum_{i \in S} \phi_i(N, q, C) = \sum_{i \in S} \phi_i(N, q', C'). \]

\(^3\)Spumon [16] does not introduce \( C' \) since the domain is unique.
This property requires the aggregate cost share of a group of agents consuming goods which are perfect substitutes to depend only on their aggregate consumption. Therefore, it implies that agents in $S$ do not improve if they reshuffle their consumptions.

**Theorem 2.** The Aumann-Shapley method is the only cost-sharing method that satisfies monotonicity, zero independence and no reshuffling.

*Proof.* The same proof as in Corollary 1 of Sprumont [16].

On the other hand, in a different approach Sprumont [16] considers the case in which agents are allowed to consume several goods.

An extended consumption profile is a list $(N, M, q)$, where $N \in \mathcal{N}$ is the set of agents, $M \in \mathcal{N}$ is the set of goods, and $q \in (\mathbb{N}^M)^N$ is the list of consumptions of the agents in $N$. Agent $i$'s consumption is now a bundle of goods $q^i \in \mathbb{N}^M$; the real number $q^i_h$ denotes his consumption of good $h \in M$, and $q^i$ is the list of consumptions of good $h$ by the agents in $N$. Write $q(N) = \sum_{i \in N} q^i$, that is $q(N) \in \mathbb{N}^M$ is the bundle of total consumption demands of all the agents. If $S \subseteq N$, $q^S \in (\mathbb{N}^M)^S$ denotes the restriction of $q$ to $S$. Given $i \in N$, write $M_i(q) = \{h \in M : q^i_h \neq 0\}$.

An extended cost-sharing problem is a list $(N, M, q, C)$, where $(N, M, q)$ is an extended consumption profile, and $C \in C(M, q(N))$ is a cost function. The set of all extended problems is denoted $\mathcal{P}^*$. An extended cost-sharing method is a mapping $\phi^*$ that assigns to every cost-sharing problem $(N, M, q, C) \in \mathcal{P}^*$ a vector $\phi^*(N, M, q, C) \in \mathbb{R}^N_+$ satisfying $\sum_{i \in N} \phi^*_i(N, M, q, C) = C(q(N))$.

The extended Aumann-Shapley method, defined by Sprumont [16], assigns to every $(N, M, q, C) \in \mathcal{P}^*$ the vector $\phi^{*\text{A}sh}(N, M, q, C)$ defined for all $i \in N$ by

$$\phi^{*\text{A}sh}_i(N, M, q, C) = \sum_{h \in M} \frac{\phi^{*\text{A}sh}_h(M, q(N), C)}{\sum_{j \in N} q^j_h} q^i_h.$$ 

The axioms to characterize this extended Aumann-Shapley value are the following ones. The first one is an adaptation of the monotonicity axiom to this context, and for the other two see Sprumont [16] (see also Albizuri [1]).

**Monotonicity**$: Let $(N, M, q, C), (N, M, q', C') \in \mathcal{P}^*$, $i \in N$ and $h \in M_i(q)$. If for all $x \in [0, q^i(N)]$ such that $x_h < q^i_h$ it holds $\partial_h C(x) \leq \partial_h C'(x)$, then

$$\phi^*_i(N, M, q, C) \leq \phi^*_i(N, M, q, C').$$

According to this axiom, if the marginal costs with respect to the goods which are consumed by agents increase, so do the cost shares of the agents.
In the following axiom, similarly as in the first variation, given $C \in C^1_T(M, q(N))$, there exists $C' \in C^1_T(M, q'(N))$ which has associated the same $c$ corresponding to $C$.

**No Reshuffling**: Let $N \in \mathcal{N}$, $M \in \mathcal{M}$, $q, q' \in (\mathbb{N}^M)^N$, $S \subseteq N$, $T \subseteq M$ and $C \in C^1_T(M, q(N))$. If

$$\sum_{i \in S} \sum_{h \in T} q^i_h = \sum_{i \in S} \sum_{h \in T} (q')^i_h \cdot q^S_{M \setminus T} = (q')^S_{M \setminus T} \text{ and } q^{N \setminus S} = (q')^{N \setminus S},$$

then

$$\sum_{i \in S} \phi^*_i(N, M, q, C) = \sum_{i \in S} \phi^*_i(N, M, q', C').$$

This axiom is a reformulation of no reshuffling given by Spumont [16] and is motivated as the original no reshuffling. This axiom relates two extended cost-sharing problems $(N, M, q, C)$ and $(N, M, q', C')$ in which goods in $T \subseteq M$ are perfect substitutes and agents in $S \subseteq N$ have the same aggregate consumption of these goods. In both problems agents in $S$ have the same consumption of goods outside $T$, and agents who are not in $S$ have the same consumption of any good. The axiom requires that agents in $S$ have the same aggregate cost share in both problems.

**Zero Independence**: Let $(N, M, q, C) \in \mathcal{P}^*$, and $i \in N$, $h \in M$. If $q^i = 0$ then $\phi^*_i(N, M, q, C) = 0$ and $\phi^*_{N \setminus i}(N, M, q, C) = \phi^*(N \setminus i, M, q^{N \setminus i}, C)$. If $q_h = 0$ then $\phi^*(N, M \setminus h, q^{M \setminus h}, C_{M \setminus h}) = \phi^*(N, M, q, C)$.

With this generalization of zero independence introduced by Sprumont [16] and the above two axioms we get the following axiomatization.

**Theorem 3.** The extended Aumann-Shapley method is the only cost-sharing method that satisfies monotonicity*, zero independence* and no reshuffling*.

**Proof.** It is clear that the extended Aumann-Shapley method satisfies the three axioms. The converse can be proved as Sprumont [16] does Corollary 2, but now using Theorem 2 instead of Corollary 1 in Sprumont [16].

5. **Conclusion**

In this paper we offer an axiomatization of the Aumann-Shapley method in the discrete case by employing monotonicity. In doing so, we have filled the gap comparing with cost-sharing problems in which goods are perfectly divisible. At the same time we have found that weaker axioms are needed in the discrete case. We have seen it by looking at the characterization of the Aumann-Shapley price mechanism introduced by Monderer and Neyman [10]. It is an open question if
axioms could be weakened in the latter case. In particular, if symmetry is not needed as an explicit requirement.

6. References


